GINZBURG-LANDAU DYNAMICS WITH A TIME-DEPENDENT MAGNETIC FIELD

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Abstract. The time-dependent Ginzburg-Landau equations of superconductivity define a dynamical process when the applied magnetic field varies with time. Sufficient conditions (in terms of the time rate of change of the applied field) are given that, if satisfied, guarantee that this dynamical process is asymptotically autonomous. As time goes to infinity, the dynamical process asymptotically approaches a dynamical system whose attractor coincides with the omega-limit set of the dynamical process.

Keywords. Ginzburg-Landau equations, superconductivity, asymptotically autonomous dynamical process, global attractor.

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1 Introduction

This article is concerned with the time evolution of the superconducting and electromagnetic properties of a superconductor in a time-dependent magnetic field, as described by the time-dependent Ginzburg-Landau (TDGL) equations of superconductivity. The dynamics of the TDGL equations were the focus of investigation in our earlier article [1], where we showed that the equations define a dynamical process in a suitably chosen Hilbert space. Here, we study the particular case of an *asymptotically autonomous dynamical process*, which arises when the time rate of change of the applied magnetic field decays sufficiently fast as time goes to infinity. We show that the dynamical process asymptotically approaches a dynamical system and that its attractor coincides with the attractor of the limiting dynamical system.

In the remainder of this section, we introduce the TDGL model of superconductivity. In Section 2, we give its functional formulation. In Section 3, we state our results; the proofs are given in Section 4.

1.1 Ginzburg-Landau Model of Superconductivity

The TDGL equations of superconductivity are

$$\eta \left(\frac{\partial}{\partial t} + i\kappa\phi \right) \psi = - \left(\frac{i}{\kappa}\nabla + \boldsymbol{A} \right)^2 \psi + \left(1 - |\psi|^2 \right) \psi, \tag{1.1}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi = -\nabla \times \nabla \times \boldsymbol{A} + \boldsymbol{J}_s + \nabla \times \boldsymbol{H}, \tag{1.2}$$

where

$$\boldsymbol{J}_s \equiv \boldsymbol{J}_s(\psi, \boldsymbol{A}) = \frac{1}{2i\kappa} \left(\psi^* \nabla\psi - \psi\nabla\psi^* \right) - |\psi|^2 \boldsymbol{A} = -\mathrm{Re} \left[\psi^* \left(\frac{i}{\kappa}\nabla + \boldsymbol{A} \right) \psi \right]. \tag{1.3}$$

The unknowns are the complex-valued *order parameter* ψ, the vector-valued *vector potential* $\boldsymbol{A}$, and the real-valued *scalar potential* ϕ. They determine the physically relevant variables, namely, the *supercurrent density* $\boldsymbol{J}_s$, the *magnetic induction* $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and the *electric field* $\boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla\phi$. The vector $\boldsymbol{H}$ represents the (externally) *applied magnetic field*; it is a given function of space and time, which is divergence free, $\nabla \cdot \boldsymbol{H} = 0$, at all times. The Eqs. (1.1) and (1.2) must be satisfied everywhere in the superconductor, at all times t, and their solution must satisfy the boundary conditions

$$\boldsymbol{n} \cdot \left(\frac{i}{\kappa}\nabla + \boldsymbol{A} \right) \psi + \frac{i}{\kappa}\gamma\psi = 0 \quad \text{and} \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A} - \boldsymbol{H}) = \boldsymbol{0}. \tag{1.4}$$

The vector $\boldsymbol{n}$ is the unit outward normal, and γ is a nonnegative function.

The parameters η and κ are the (dimensionless) friction coefficient and Ginzburg-Landau parameter, respectively. As usual, $\nabla \equiv$ grad, $\nabla\times \equiv$ curl, $\nabla\cdot \equiv$ div, and $\nabla^2 = \nabla \cdot \nabla \equiv \Delta$; a superscript * denotes complex conjugation, and i is the imaginary unit. The symbol ∂_t denotes the partial derivative $\partial/\partial t$.

We assume that the vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ take values in $\mathbf{R}^n$ ($n = 2$ or $n = 3$), the superconductor occupies a bounded domain Ω in $\mathbf{R}^n$, and the boundary $\partial\Omega$ of Ω is of class $C^{1,1}$.

1.2 Gauge Choice

The TDGL equations are invariant under the gauge transformation

$$\mathcal{G}_\chi : (\psi, \boldsymbol{A}, \phi) \mapsto \left(\psi e^{i\kappa\chi}, \boldsymbol{A} + \nabla\chi, \phi - \partial_t\chi\right). \tag{1.5}$$

The *gauge* χ can be any (sufficiently smooth) real scalar-valued function of position and time. For the present investigation, we adopt the "$\phi = -\omega(\nabla\cdot\boldsymbol{A})$" gauge, with ω a real nonnegative parameter [2]. This gauge is determined by taking $\chi \equiv \chi_\omega(x,t)$ as a solution of the boundary-value problem

$$(\partial_t - \omega\Delta)\chi = \phi + \omega(\nabla \cdot \boldsymbol{A}) \quad \text{on } \Omega \times (0,\infty), \tag{1.6}$$

$$\omega(\boldsymbol{n} \cdot \nabla\chi) = -\omega(\boldsymbol{n} \cdot \boldsymbol{A}) \quad \text{on } \partial\Omega \times (0,\infty). \tag{1.7}$$

In this gauge, $\boldsymbol{A}$ and ϕ satisfy the identities

$$\phi + \omega(\nabla \cdot \boldsymbol{A}) = 0 \quad \text{on } \Omega \times (0,\infty), \qquad \omega(\boldsymbol{n} \cdot \boldsymbol{A}) = 0 \quad \text{on } \partial\Omega \times (0,\infty). \tag{1.8}$$

If the triple $(\psi, \boldsymbol{A}, \phi)$ satisfies the TDGL equations, then the second identity can be strengthened, and we have [1]

$$\phi + \omega(\nabla \cdot \boldsymbol{A}) = 0 \quad \text{on } \Omega \times (0,\infty), \qquad \boldsymbol{n} \cdot \boldsymbol{A} = 0 \quad \text{on } \partial\Omega \times (0,\infty). \tag{1.9}$$

The gauge choice fixes ϕ, such that $\int_\Omega \phi \, \mathrm{d}x = 0$ at all times.

In the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge, the TDGL equations reduce to

$$\eta\frac{\partial\psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \psi + i\eta\kappa\omega\psi(\nabla\cdot\boldsymbol{A}) + \left(1 - |\psi|^2\right)\psi \quad \text{in } \Omega\times(0,\infty), \tag{1.10}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \times \nabla \times \boldsymbol{A} + \omega\nabla(\nabla \cdot \boldsymbol{A}) + \boldsymbol{J}_s + \nabla \times \boldsymbol{H} \quad \text{in } \Omega \times (0,\infty), \tag{1.11}$$

where $\boldsymbol{J}_s$ is again given by Eq. (1.3), and

$$\boldsymbol{n}\cdot\nabla\psi+\gamma\psi=0, \quad \boldsymbol{n}\cdot\boldsymbol{A}=0, \quad \boldsymbol{n}\times(\nabla\times\boldsymbol{A}-\boldsymbol{H})=\boldsymbol{0} \quad \text{on } \partial\Omega\times(0,\infty). \tag{1.12}$$

Henceforth, we use the term "gauged TDGL equations" to refer to the system of Eqs. (1.10)–(1.12). The gauged TDGL equations govern the evolution of the pair $(\psi, \boldsymbol{A})$ from the initial data,

$$\psi=\psi_0 \quad \text{and} \quad \boldsymbol{A}=\boldsymbol{A}_0 \quad \text{on } \Omega\times\{0\}, \tag{1.13}$$

where ψ_0 and $\boldsymbol{A}_0$ are given.

2 Functional Formulation

In this section we reformulate the gauged TDGL equations as an abstract evolution equation in a Hilbert space. The notational conventions are established in Section 2.1; preliminary material is presented in Section 2.2; the functional formulation of the gauged TDGL equations is given in Section 2.3.

2.1 Notation

The symbol C denotes a generic positive constant, not necessarily the same at different instances. All Banach spaces are real; the (real) dual of a Banach space X is denoted by X'.

We recall that $\Omega\subset\mathbf{R}^n$ ($n=2$ or $n=3$), Ω is bounded, and its boundary $\partial\Omega$ is of class $C^{1,1}$.

The Banach spaces in this investigation are the standard ones [3, 4]: the Lebesgue spaces $L^p(\Omega)$ for $1\le p<\infty$, with norm $\|\cdot\|_{L^p(\Omega)}$; the Sobolev spaces $W^{m,2}(\Omega)$ for nonnegative integer m, with norm $\|\cdot\|_{W^{m,2}}$; the fractional Sobolev spaces $W^{s,2}(\Omega)$, with noninteger s; and the spaces $C^\nu(\Omega)$, for $\nu\ge 0$, $\nu=m+\lambda$ with $0\le\lambda<1$, of m times continuously differentiable functions on Ω, whose mth-order derivatives satisfy a Hölder condition with exponent λ if ν is not an integer, with norm $\|\cdot\|_{C^\nu}$. The inner product in $L^2(\Omega)$ is $(\cdot\,,\cdot)$, and $W^{m,2}(\Omega)$ is a Hilbert space for the inner product $(\cdot\,,\cdot)_{m,2}$, given by $(u,v)_{m,2}=\sum_{|\alpha|\le m}(\partial^\alpha u,\partial^\alpha v)$ for $u,v\in W^{m,2}(\Omega)$. The definitions extend to spaces of vector-valued functions in the usual way, with the *caveat* that the inner product in $[L^2(\Omega)]^n$ is defined by $(u,v)=\int_\Omega u\cdot v$, where $\cdot$ indicates the scalar product in $\mathbf{R}^n$. Complex-valued functions are interpreted as vector-valued functions with two real components.

Functions of space and time defined on $\Omega \times [0,T]$, for some $T > 0$, are considered as mappings from the time domain $[0,T]$ into a Banach space $X = (X, \|\cdot\|_X)$ of functions on the spatial domain Ω and may be considered as elements of $L^p(0,T;X)$ for $1 \le p \le \infty$, or $W^{s,2}(0,T;X)$ for nonnegative s, or $C^\nu(0,T;X)$ for $\nu \ge 0$.

Function spaces of ordered pairs $(\psi, \boldsymbol{A})$, where $\psi : \Omega \to \mathbf{R}^2$ and $\boldsymbol{A} : \Omega \to \mathbf{R}^n$ $(n = 2,3)$, play an important role in the study of the TDGL equations. Because the regularity requirements for ψ and $\boldsymbol{A}$ are the same, it is convenient to adopt the special notation $\mathcal{X} = X^2 \times X^n$ for any Banach space X of real-valued functions defined on Ω; X^2 and X^n are the underlying spaces for the order parameter ψ and the vector potential $\boldsymbol{A}$, respectively. A suitable framework for the functional analysis of the gauged TDGL equations is the Cartesian product

$$\mathcal{W}^{1+\alpha,2} = [W^{1+\alpha,2}(\Omega)]^2 \times [W^{1+\alpha,2}(\Omega)]^n,$$

with $\frac{1}{2} < \alpha < 1$. This space is continuously imbedded in $\mathcal{W}^{1,2} \cap \mathcal{L}^\infty$.

A *weak solution* of the gauged TDGL equations on the interval $[0,T]$, for some $T > 0$, is a function $(\psi, \boldsymbol{A}) \in C(0,T;\mathcal{W}^{1+\alpha,2})$, with values $(\psi, \boldsymbol{A})(t) \equiv (\psi(t), \boldsymbol{A}(t)) \in \mathcal{W}^{1+\alpha,2}$, which satisfies Eqs. (1.10)–(1.12) in the sense of distributions for each $t \in (0,T)$.

2.2 Reduction to Homogeneous Form

Before giving the functional formulation of the gauged TDGL equations, we reduce the boundary conditions (1.12) to homogeneous form. The reduction is done at each fixed instant; time is therefore a parameter, which we will not write explicitly.

Assume $\boldsymbol{H} \in [L^2(\Omega)]^n$, and define $\boldsymbol{A}_{\mathbf{H}}$ as a minimizer of the convex quadratic functional $J_\omega \equiv J_\omega[\boldsymbol{A}]$,

$$J_\omega[\boldsymbol{A}] = \int_\Omega \left[\omega(\nabla \cdot \boldsymbol{A})^2 + |\nabla \times \boldsymbol{A} - \boldsymbol{H}|^2\right] \mathrm{d}x, \tag{2.1}$$

on the domain

$$\mathcal{D}(J_\omega) = \{\boldsymbol{A} \in [W^{1,2}(\Omega)]^n : \boldsymbol{n} \cdot \boldsymbol{A} = 0 \text{ on } \partial\Omega\}.$$

If $\omega > 0$, this minimizer is unique, and $\nabla \cdot \boldsymbol{A}_{\mathbf{H}} = 0$ in Ω. If $\omega = 0$, we restrict the minimization to the closed linear subspace $\mathcal{D}_0(J_0) = \{\boldsymbol{A} \in \mathcal{D}(J_0) : \nabla \cdot \boldsymbol{A} = 0 \text{ in } \Omega\}$ of $\mathcal{D}(J_0)$, where J_0 has a unique minimizer $\boldsymbol{A}_{\mathbf{H}}$; see [1, Lemma 3]. In either case, $\boldsymbol{A}_{\mathbf{H}}$ is the unique solution of the boundary-value problem

$$\nabla \times \nabla \times \boldsymbol{A}_{\mathbf{H}} = \nabla \times \boldsymbol{H} \quad \text{and} \quad \nabla \cdot \boldsymbol{A}_{\mathbf{H}} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{n} \cdot \boldsymbol{A}_{\mathbf{H}} = 0 \quad \text{and} \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A}_{\mathbf{H}} - \boldsymbol{H}) = \mathbf{0} \quad \text{on } \partial\Omega. \tag{2.3}$$

Thus, $\boldsymbol{A}_{\mathbf{H}}$ takes care of the inhomogeneity in the boundary conditions (1.12). The mapping $\boldsymbol{H} \mapsto \boldsymbol{A}_{\mathbf{H}}$, which is linear and time independent, is continuous from $[W^{\theta,2}(\Omega)]^n$ to $[W^{1+\theta,2}(\Omega)]^n$ for $0 \le \theta \le 1$; see [1, Lemma 4].

The boundary conditions in the gauged TDGL equations become homogeneous if we formulate the equations in terms of ψ and the *reduced vector potential* $\boldsymbol{A}'$,

$$\boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}}. \tag{2.4}$$

In fact, we may summarize the gauged TDGL equations in the form

$$\frac{\partial \psi}{\partial t} - \frac{1}{\eta\kappa^2}\Delta\psi = \varphi \quad \text{in } \Omega \times (0, \infty), \tag{2.5}$$

$$\frac{\partial \boldsymbol{A}'}{\partial t} + \nabla \times \nabla \times \boldsymbol{A}' - \omega\nabla(\nabla \cdot \boldsymbol{A}') = \boldsymbol{F} \quad \text{in } \Omega \times (0, \infty), \tag{2.6}$$

$$\boldsymbol{n} \cdot \nabla\psi + \gamma\psi = 0, \quad \boldsymbol{n} \cdot \boldsymbol{A}' = 0, \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A}') = \mathbf{0} \quad \text{on } \partial\Omega \times (0, \infty), \tag{2.7}$$

where φ and $\boldsymbol{F}$ are nonlinear functions of ψ and $\boldsymbol{A}'$,

$$\varphi \equiv \varphi(t, \psi, \boldsymbol{A}') = \frac{1}{\eta}\Big[-\frac{2i}{\kappa}(\nabla\psi) \cdot (\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}})$$

$$-\frac{i}{\kappa}(1 - \eta\kappa^2\omega)\psi(\nabla \cdot \boldsymbol{A}') - \psi|\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}|^2 + \left(1 - |\psi|^2\right)\psi\Big], \tag{2.8}$$

$$\boldsymbol{F} \equiv \boldsymbol{F}(t, \psi, \boldsymbol{A}') = \boldsymbol{J}_s' - |\psi|^2\boldsymbol{A}_{\mathbf{H}} - \boldsymbol{A}_{\partial_t\mathbf{H}}. \tag{2.9}$$

Here we have used the abbreviation $\boldsymbol{J}_s' = \boldsymbol{J}_s(\psi, \boldsymbol{A}')$, where $\boldsymbol{J}_s$ is the expression for the supercurrent density, given by Eq. (1.3). The equations are supplemented by initial data, which follow from Eqs. (1.13) and (2.4),

$$\psi = \psi_0 \quad \text{and} \quad \boldsymbol{A}' = \boldsymbol{A}_0 - \boldsymbol{A}_{\mathbf{H}} \quad \text{on } \Omega \times \{0\}. \tag{2.10}$$

2.3 Gauged TDGL Equations

From here on, the analysis is restricted to the case $\omega > 0$.

Let the vector $u : [0, \infty) \to \mathcal{L}^2$ represent the pair $(\psi, \boldsymbol{A}')$,

$$u = (\psi, \boldsymbol{A}') \equiv (\psi, \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}}), \tag{2.11}$$

and let $\mathcal{A}$ be the linear selfadjoint operator in $\mathcal{L}^2$ associated with the quadratic form $Q_\omega \equiv Q_\omega[u]$,

$$Q_\omega[u] = \int_\Omega \left[\frac{1}{\eta\kappa^2} |\nabla\psi|^2 + \omega(\nabla \cdot \boldsymbol{A}')^2 + |\nabla \times \boldsymbol{A}'|^2 \right] \mathrm{d}x + \int_{\partial\Omega} \frac{\gamma}{\eta\kappa^2} |\psi|^2 \, \mathrm{d}\sigma(x), \tag{2.12}$$

on the domain

$$\mathcal{D}(Q_\omega) = \mathcal{D}(\mathcal{A}^{1/2}) = \{u = (\psi, \boldsymbol{A}') \in \mathcal{W}^{1,2} : \boldsymbol{n} \cdot \boldsymbol{A}' = 0 \text{ on } \partial\Omega\}.$$

The quadratic form Q_ω is nonnegative. Furthermore, since $\omega > 0$, $Q_\omega[\psi, \boldsymbol{A}'] + c\|\psi\|_{L^2}$ is coercive on $\mathcal{W}^{1,2}$ for any constant $c > 0$. Hence, $\mathcal{A}$ is positive definite in $\mathcal{L}^2$ [5, Chapter I, Eq. (5.45)]. If no confusion is possible, we use the same symbol $\mathcal{A}$ for the restrictions $\mathcal{A}_\psi$ and $\mathcal{A}_{\mathbf{A}'}$ of $\mathcal{A}$ to the respective linear subspaces $[L^2(\Omega)]^2 \equiv [L^2(\Omega)]^2 \times \{0\}$ (for ψ) and $[L^2(\Omega)]^n \equiv \{0\} \times [L^2(\Omega)]^n$ (for $\boldsymbol{A}'$) of $\mathcal{L}^2$.

A weak solution of the boundary-value problem (2.5)–(2.7) that satisfies the initial conditions (2.10) corresponds to a *mild solution* $u \in \mathcal{L}^2$ of the initial-value problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u = \mathcal{F}(t, u(t)) \quad \text{for } t > 0; \quad u(0) = u_0, \tag{2.13}$$

where $\mathcal{F}(t, u) = (\varphi, \boldsymbol{F})$, φ and $\boldsymbol{F}$ given by Eqs. (2.8) and (2.9), and $u_0 = (\psi_0, \boldsymbol{A}'_0)$. With $\frac{1}{2} < \alpha < 1$ and $u_0 \in \mathcal{W}^{1+\alpha,2}$, a mild solution of (2.13) on $[0, T]$ is a continuous function $u : [0, T] \to \mathcal{W}^{1+\alpha,2}$, such that

$$u(t) = \mathrm{e}^{-\mathcal{A}t} u_0 + \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)} \mathcal{F}(s, u(s)) \, \mathrm{d}s \quad \text{for } 0 \le t \le T. \tag{2.14}$$

The equation $\mathcal{A}u = f$ in $\mathcal{L}^2$, where $f = (\varphi, \boldsymbol{F})$ is any element of $\mathcal{L}^2$, is equivalent to a system of uncoupled boundary-value problems,

$$-\frac{1}{\eta\kappa^2} \Delta\psi = \varphi \quad \text{in } \Omega, \quad \boldsymbol{n} \cdot \nabla\psi + \gamma\psi = 0 \quad \text{on } \partial\Omega; \tag{2.15}$$

$$\nabla \times \nabla \times \boldsymbol{A}' - \omega\nabla(\nabla \cdot \boldsymbol{A}') = \boldsymbol{F} \quad \text{in } \Omega, \quad \boldsymbol{n} \cdot \boldsymbol{A}' = 0, \ \boldsymbol{n} \times (\nabla \times \boldsymbol{A}') = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{2.16}$$

(More precisely, the system of Eqs. (2.15)–(2.16) holds in the dual space $\mathcal{D}(Q_\omega)'$ of $\mathcal{D}(Q_\omega)$ with respect to the inner product in $\mathcal{L}^2$.) Boundary-value problems of this type have been studied by Georgescu [6]. Applying his results, we conclude that $\mathcal{D}(\mathcal{A})$ is a closed linear subspace of $\mathcal{W}^{2,2}$. Since $\mathcal{A}$ is positive definite on $\mathcal{L}^2$, its fractional powers $\mathcal{A}^\theta$ are well defined for all $\theta \in \mathbf{R}$; they are unbounded for $\theta > 0$. Interpolation theory shows that $\mathcal{D}(\mathcal{A}^\theta)$ is a closed linear subspace of $\mathcal{W}^{2\theta,2}$ for $0 < \theta < 1$.

3 Results

In this section we present the results of our investigation in the form of three theorems and a corollary. The proofs of the theorems are given in Section 4. We begin by recalling some relevant results from our earlier article [1].

The TDGL equations generate a dynamical process if the data satisfy the following hypotheses:

(H1) $\Omega \subset \mathbf{R}^n$ ($n = 2$ or 3) is bounded, with $\partial\Omega$ of class $C^{1,1}$—that is, $\partial\Omega$ is a compact $(n-1)$-manifold described by Lipschitz-continuously differentiable charts;

(H2) $\gamma : \partial\Omega \to \mathbf{R}$ is Lipschitz continuous, with $\gamma(x) \geq 0$ for all $x \in \partial\Omega$;

(H3) $\omega, \alpha, \beta \in \mathbf{R}$ are constants, such that $0 < \omega < \infty$, $\frac{1}{2} < \alpha < 1$, and $0 \leq \beta < \frac{1}{2}(1-\alpha)$; and

(H4) $\boldsymbol{H} \in L^\infty(0,T;[W^{\alpha,2}(\Omega)]^n) \cap W^{1,2}(0,T;[L^2(\Omega)]^n)$ for any $T \in (0,\infty)$.

The initial-value problem (2.13) has a unique mild solution $u \in C(0,T;\mathcal{W}^{1+\alpha,2})$ for any $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ and any $T > 0$ [1, Theorem 1]. These mild solutions generate the *dynamical process* $U = \{U(t,s) : 0 \leq s \leq t \leq T\}$ on $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ by the definition

$$u(t) = U(t,s)u(s), \quad 0 \leq s \leq t \leq T. \tag{3.1}$$

The process U completely describes the dynamics of the TDGL equations. In the present article we focus on the large-time asymptotic behavior of $U(t,s)$ as $t \to \infty$ (s fixed, $s \geq 0$) in the special case where the applied magnetic field is *asymptotically stationary.*

3.1 Asymptotically Stationary Field

Instead of the hypothesis **(H4)**, we impose the stronger hypotheses

(H4') $\boldsymbol{H} \in L^{\infty}(0,\infty;[W^{\alpha',2}(\Omega)]^n)$ for some $\alpha' \in (\alpha, 1)$, and

(H4") $\partial_t \boldsymbol{H} \in L^1(0,\infty;[L^2(\Omega)]^n) \cap L^2(0,\infty;[L^2(\Omega)]^n)$.

We claim that, under these hypotheses, the applied magnetic field $\boldsymbol{H}$ approaches a limit in $[W^{\alpha,2}(\Omega)]^n$ as $t \to \infty$.

For any $t \geq s \geq 0$, we have

$$\boldsymbol{H}(t) = \boldsymbol{H}(s) + \int_s^t \partial_t \boldsymbol{H}(\cdot\,, t')\,\mathrm{d}t'. \tag{3.2}$$

The integral exists as a Bochner integral in $[L^2(\Omega)]^n$. The hypothesis **(H4")** guarantees that

$$\int_0^{\infty} \left(\int_{\Omega} |\partial_t \boldsymbol{H}(x,t')|^2 \,\mathrm{d}x \right)^{1/2} \mathrm{d}t' < \infty,$$

so the limit

$$\boldsymbol{H}_{\infty} = \lim_{t\to\infty} \boldsymbol{H}(t) \tag{3.3}$$

exists in $[L^2(\Omega)]^n$ and is given by

$$\boldsymbol{H}_{\infty} = \boldsymbol{H}(s) + \int_s^{\infty} \partial_t \boldsymbol{H}(\cdot\,, t')\,\mathrm{d}t', \quad s \geq 0. \tag{3.4}$$

Combining Eq. (3.3) with the hypothesis **(H4')**, we obtain the same limiting relation (3.3) in the weak topology on $[W^{\alpha',2}(\Omega)]^n$ and, consequently, in the strong topology on $[W^{\alpha,2}(\Omega)]^n$, because the imbedding $W^{\alpha',2}(\Omega) \hookrightarrow W^{\alpha,2}(\Omega)$ is compact for $\alpha < \alpha'$, by Rellich's theorem and interpolation.

3.2 Large-Time Asymptotic Behavior

Given the limiting relation (3.3), we compare the large-time asymptotic behavior of the solution of the gauged TDGL equations, described by the dynamical process U, with that of the gauged TDGL equations for a superconductor in the stationary applied magnetic field $\boldsymbol{H}_{\infty}$,

$$\eta \frac{\partial \psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \psi + i\eta\kappa\omega\psi(\nabla\cdot\boldsymbol{A}) + \left(1 - |\psi|^2\right)\psi \quad \text{in } \Omega\times(0,\infty), \tag{3.5}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \times \nabla \times \boldsymbol{A} + \omega \nabla(\nabla \cdot \boldsymbol{A}) + \boldsymbol{J}_s + \nabla \times \boldsymbol{H}_\infty \quad \text{in } \Omega \times (0, \infty), \quad (3.6)$$

$$\boldsymbol{n} \cdot \nabla \psi + \gamma \psi = 0, \quad \boldsymbol{n} \cdot \boldsymbol{A} = 0, \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A} - \boldsymbol{H}_\infty) = \boldsymbol{0} \quad \text{on } \partial\Omega \times (0, \infty). \quad (3.7)$$

The quantity $\boldsymbol{J}_s$ is given in terms of ψ and $\boldsymbol{A}$ by the same expression (1.3) as in the time-dependent field case.

Equations (3.5)–(3.7) define a dynamical system [1]. Before we can introduce this dynamical system, we must homogenize the boundary conditions (3.7). The homogenization is achieved in the usual way by reformulating Eqs. (3.5)–(3.7) in terms of ψ and a reduced vector potential $\boldsymbol{A}'$,

$$\boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}_\infty}. \quad (3.8)$$

Here, $\boldsymbol{A}_{\mathbf{H}_\infty}$ is the (unique) solution of the boundary-value problem

$$\nabla \times \nabla \times \boldsymbol{A}_{\mathbf{H}_\infty} = \nabla \times \boldsymbol{H}_\infty \quad \text{and} \quad \nabla \cdot \boldsymbol{A}_{\mathbf{H}_\infty} = 0 \quad \text{in } \Omega, \quad (3.9)$$

$$\boldsymbol{n} \cdot \boldsymbol{A}_{\mathbf{H}_\infty} = 0 \quad \text{and} \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A}_{\mathbf{H}_\infty} - \boldsymbol{H}_\infty) = \boldsymbol{0} \quad \text{on } \partial\Omega. \quad (3.10)$$

Equations (3.5)–(3.7) correspond to the abstract initial-value problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \mathcal{A}v = \mathcal{G}(v(t)) \quad \text{for } t > 0; \quad v(0) = v_0, \quad (3.11)$$

for a vector $v : [0, \infty) \to \mathcal{L}^2$, whose components are ψ and $\boldsymbol{A}'$,

$$v = (\psi, \boldsymbol{A}') \equiv (\psi, \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}_\infty}). \quad (3.12)$$

The nonlinear function $\mathcal{G}$ in Eq. (3.11) stands for the vector $\mathcal{G}(v) = (\chi, \boldsymbol{G})$, where

$$\chi \equiv \chi(v) = \frac{1}{\eta}\Big[-\frac{2i}{\kappa}(\nabla\psi) \cdot (\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}_\infty})$$

$$-\frac{i}{\kappa}(1 - \eta\kappa^2\omega)\psi(\nabla \cdot \boldsymbol{A}') - \psi|\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}_\infty}|^2 + \left(1 - |\psi|^2\right)\psi\Big], \quad (3.13)$$

$$\boldsymbol{G} \equiv \boldsymbol{G}(v) = \boldsymbol{J}_s' - |\psi|^2 \boldsymbol{A}_{\mathbf{H}_\infty}. \quad (3.14)$$

Here, $\boldsymbol{J}_s' = \boldsymbol{J}_s(\psi, \boldsymbol{A}')$, as in Eq. (2.9). The vector $v_0 = (\psi_0, \boldsymbol{A}_0 - \boldsymbol{A}_{\mathbf{H}_\infty})$ is given, $v_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$. The solutions of the abstract initial-value problem (3.11) generate a *dynamical system* $S = \{S(t) : t \geq 0\}$ on $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ by the definition

$$v(t) = S(t)v_0, \quad t \geq 0; \quad (3.15)$$

see [1, Corollary 2]. The system S completely describes the dynamics of the TDGL equations (3.5)–(3.7).

The first theorem describes how the large-time asymptotic behavior of $U(t, s)$ (s fixed, $s \geq 0$) compares with that of $S(t)$ as $t \to \infty$.

Theorem 1 *Let ε and R be arbitrary positive numbers, and let B_R be the ball of radius R centered at the origin in $\mathcal{W}^{1+\alpha,2}$. There exist numbers $\delta > 0$, $\Lambda \geq 0$, and $t_0 \geq 0$ such that, for any $u_s, v_s \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2}) \cap B_R$ satisfying the inequality $\|u_s - v_s\|_{\mathcal{W}^{1+\alpha,2}} \leq \delta$, we have*

$$\|U(t,s)u_s - S(t-s)v_s\|_{\mathcal{W}^{1+\alpha,2}} \leq \varepsilon \ \mathrm{e}^{\Lambda(t-s)}, \tag{3.16}$$

for all $s, t \in \mathbf{R}$ with $t_0 \leq s \leq t < \infty$.

The proof of Theorem 1 is given in Section 4.1.

Theorem 1 shows that the dynamical process U is *asymptotically autonomous*; see, for example, [8, Section 3.7, p. 46]. A dynamical process $U = \{U(t,s) : 0 \leq s \leq t < \infty\}$ on a Banach space $\mathcal{X}$ is asymptotically autonomous if there exists a dynamical system $S = \{S(t) : t \geq 0\}$ on $\mathcal{X}$ with the following property: For arbitrary positive numbers ε, R, and T, there exists some $t_0 \geq 0$ such that

$$\|U(s+t,s)u_0 - S(t)u_0\|_{\mathcal{X}} \leq \varepsilon \tag{3.17}$$

for all $u_0 \in \mathcal{X}$ with $\|u_0\|_{\mathcal{X}} \leq R$ and for all $(s,t) \in [t_0, \infty) \times [0,T]$. Equivalently: For all positive numbers R and T,

$$\|U(s+t,s)u_0 - S(t)u_0\|_{\mathcal{X}} \to 0 \quad \text{as } s \to \infty, \tag{3.18}$$

uniformly in (u_0, t), for all $u_0 \in \mathcal{X}$ with $\|u_0\|_{\mathcal{X}} \leq R$ and all $t \in [0,T]$.

Corollary 1 *The dynamical process $U = \{U(t,s) : 0 \leq s \leq t < \infty\}$ defined in Eq. (3.1) is asymptotically autonomous; its large-time asymptotic limit is the dynamical system $S = \{S(t) : t \geq 0\}$ defined in Eq. (3.15). Moreover, if ε, R, and T are arbitrary positive numbers, then there exist numbers $\delta > 0$ and $t_0 \geq 0$ such that, for any $u_s, v_s \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2}) \cap B_R$ satisfying the inequality $\|u_s - v_s\|_{\mathcal{W}^{1+\alpha,2}} \leq \delta$, we have*

$$\|U(s+t,s)u_s - S(t)v_s\|_{\mathcal{W}^{1+\alpha,2}} \leq \varepsilon, \tag{3.19}$$

for all $s, t \in \mathbf{R}$ with $t_0 \leq s < \infty$ and $0 \leq t \leq T$.

Following [1], we introduce the energy-type functional

$$E_{\omega,\mathbf{H}(t)}[\psi, \boldsymbol{A}] = \int_\Omega \left[\left| \left(\frac{i}{\kappa}\nabla + \boldsymbol{A} \right) \psi \right|^2 + \frac{1}{2}\left(1 - |\psi|^2\right)^2 + 2\omega(\nabla \cdot \boldsymbol{A})^2 \right.$$

$$\left. + |\nabla \times \boldsymbol{A} - \boldsymbol{H}(t)|^2 \right] \mathrm{d}x + \int_{\partial\Omega} \gamma \left| \frac{i}{\kappa}\psi \right|^2 \mathrm{d}\sigma(x) \tag{3.20}$$

and its analog with $\boldsymbol{H}(t)$ replaced by $\boldsymbol{H}_\infty$. Since $\boldsymbol{H}$ is time dependent, $E_{\omega,\mathbf{H}(t)}$ is *not* a Liapunov functional for U. However, a slight modification of $E_{\omega,\mathbf{H}(t)}$ gets us closer to a Liapunov functional. Indeed, from [1, Lemma 1] we have the inequality

$$E^{1/2}_{\omega,\mathbf{H}(t)} - P(t) \leq E^{1/2}_{\omega,\mathbf{H}(s)} - P(s) \quad \text{for } 0 \leq s \leq t < \infty, \tag{3.21}$$

where the function P, defined by the expression

$$P(t) = \int_0^t \left(\int_\Omega |\partial_t \boldsymbol{H}(x,t')|^2 \, \mathrm{d}x \right)^{1/2} \mathrm{d}t' \quad \text{for } t \geq 0, \tag{3.22}$$

is bounded for all times, because of the hypothesis **(H4")**. Hence, the quantity $\left(E^{1/2}_{\omega,\mathbf{H}(t)} - P(t)\right)^2$ plays the role of a Liapunov functional.

Theorem 2 *The dynamical process U defined in Eq. (3.1) and the dynamical system S defined in Eq. (3.15) have the following properties:*

(i) *$E_{\omega,\mathbf{H}_\infty}$ is a Liapunov functional for S in the sense of [7, Chapter VII, Definition 4.1].*

(ii) *The functional $E_{\omega,\mathbf{H}(t)}$ satisfies the inequality (3.21), where the function P, defined in Eq. (3.22), is bounded for all times.*

(iii) *The orbit of each $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ under $U(s+t,s)$ (s fixed, $s \geq 0$) and $S(t)$, $t \geq 0$, has compact closure in $\mathcal{W}^{1+\alpha,2}$.*

(iv) *The omega-limit set of each $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ under $U(s+t,s)$ (s fixed, $s \geq 0$) and $S(t)$, $t \geq 0$, is a nonempty compact connected set of divergence-free equilibria for S.*

The proof of Theorem 2 is given in Section 4.2.

We need to distinguish the omega-limit sets for the dynamical process $U(s+t,s)$ (s fixed, $s \geq 0$) from those for the dynamical system $S(t)$ as $t \to \infty$. We do so by subscripting the former, which depend on the choice of s. Thus, for $s \geq 0$ and $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ fixed, we denote the omega-limit set of the orbit $\{U(s+t,s)u_0 : t \geq 0\}$ in $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ by $\omega_s(u_0)$,

$$\omega_s(u_0) = \bigcap_{t_0 \geq 0} \overline{\{U(s+t,s)u_0 : t \geq t_0\}},$$

where the closure is taken in $\mathcal{W}^{1+\alpha,2}$, and keep the notation $\omega(u_0)$ for the omega-limit set of the orbit $\{S(t)u_0 : t \geq 0\}$. It follows from the identity $U(\tau,t)U(t,s) = U(\tau,s)$ for $0 \leq s \leq t \leq \tau < \infty$ that $\omega_t(U(t,s)u_0) = \omega_s(u_0)$ whenever $0 \leq s \leq t < \infty$ and $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$.

An *attractor* for the dynamical process U is the omega-limit set of one of its open neighborhoods B in $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$,

$$\omega(B;U) = \bigcap_{t_0 \geq 0} \overline{\bigcup_{s \geq 0,\, t \geq t_0} U(s+t,s)B},$$

where the closure is taken in $\mathcal{W}^{1+\alpha,2}$. An attractor is called a *global attractor* if it attracts all its open bounded neighborhoods. Notice that, for the dynamical system S,

$$\omega(B;S) = \bigcap_{t_0 \geq 0} \overline{\bigcup_{t \geq t_0} S(t)B}.$$

The existence of a global attractor $\mathcal{A}_S$ for S follows from [1, Corollary 2 and Theorem 3]; see [8, Theorem 3.4.8] and [9, Theorem 4.4]. The existence of a global attractor $\mathcal{A}_U$ for U follows from [1, Corollary 2] and Theorem 2 of the present article; see [8, Theorem 3.7.2] and [9, Theorem 4.4].

Our final theorem shows that the dynamical process U and the dynamical system S have the same global attractor.

Theorem 3 *The dynamical process $U = \{U(s+t,s) : s,t \geq 0\}$ has a global attractor, $\mathcal{A}_U$; the dynamical system $S = \{S(t) : t \geq 0\}$ has a global attractor, $\mathcal{A}_S$. The two global attractors coincide. If the set $\mathcal{E}$ of all stationary points of S is discrete, then $\mathcal{A}_U = \mathcal{A}_S$ is the union of $\mathcal{E}$ and the heteroclinic orbits between points of $\mathcal{E}$.*

The proof of Theorem 3 is given in Section 4.3.

4 Proofs

Before we give the proofs of the theorems, we recall some general properties of the fractional powers of the operator $\mathcal{A}$ defined in Eq. (2.12) and the semigroup generated by $-\mathcal{A}$; cf. [4].

The fractional powers $\mathcal{A}^\theta$ of the second-order elliptic differential operator $\mathcal{A}$ defined in Eq. (2.12) are well defined for all real θ. They are unbounded for $\theta > 0$. The domain $\mathcal{D}(\mathcal{A}^\theta)$ is a closed linear subspace of $\mathcal{W}^{2\theta,2}$ for $0 < \theta < 1$; hence, $C^\beta(0,T;\mathcal{D}(\mathcal{A}^\theta))$ is a closed linear subspace of $C^\beta(0,T;\mathcal{W}^{2\theta,2})$ for this range of values of θ. Furthermore, for $\frac{3}{2} < \theta \le 2$ (and $n = 2$ or 3), the traces of $\nabla\psi$, $\boldsymbol{A}$, and $\nabla\times\boldsymbol{A}$ belong to the spaces $[W^{\theta-3/2,2}(\partial\Omega)]^{2n}$, $[W^{\theta-1/2,2}(\partial\Omega)]^n$, and $[W^{\theta-3/2,2}(\partial\Omega)]^n$, respectively, and satisfy the boundary conditions specified in Eqs. (2.15) and (2.16). Similarly, the applied vector potential $\boldsymbol{A}_{\mathbf{H}}$ and its curl $\nabla\times\boldsymbol{A}_{\mathbf{H}}$ satisfy the boundary conditions (2.3) if $\boldsymbol{H} \in [W^{\theta-1,2}(\Omega)]^n$.

The semigroup generated by $-\mathcal{A}$ satisfies the inequality

$$\|\mathcal{A}^{\alpha/2}\mathrm{e}^{-\mathcal{A}s}\|_{\mathcal{L}^2} \le C\max\{s^{-\alpha/2},1\}\,\mathrm{e}^{-\lambda_1 s} \quad \text{for } 0 < s < \infty, \tag{4.1}$$

where the positive constant C does not depend on s and λ_1 denotes the first (smallest) eigenvalue of $\mathcal{A}$ in $\mathcal{L}^2$; see [4, Theorem 1.4.3]. Note that $\lambda_1 \ge 0$.

4.1 Proof of Theorem 1

Proof. The proof is based on Gronwall's lemma aplied to the initial-value problem (4.4) in the space $C([s,\infty);\mathcal{W}^{1+\alpha,2})$ for s fixed, $s \ge 0$.

Given any fixed $s \ge 0$ and any two vectors $u_s, v_s \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$, we set $u(t) = U(t,s)u_s$ and $v(t) = S(t-s)v_s$ for all $t \ge s$. Thus, u and v are the (unique) mild solutions of the initial-value problems

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u = \mathcal{F}(t,u(t)) \quad \text{for } t > s; \quad u(s) = u_s, \tag{4.2}$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \mathcal{A}v = \mathcal{G}(v(t)) \quad \text{for } t > s; \quad v(s) = v_s, \tag{4.3}$$

in $\mathcal{L}^2$, respectively. Here, $\mathcal{F}(t,u) = (\varphi, \boldsymbol{F})$, with φ and $\boldsymbol{F}$ given by Eqs. (2.8) and (2.9), and $\mathcal{G}(v) = (\chi, \boldsymbol{G})$, with χ and $\boldsymbol{G}$ given by Eqs. (3.13) and (3.14).

We set $\boldsymbol{h}(t) = \boldsymbol{H}(t) - \boldsymbol{H}_\infty$ and $\boldsymbol{A}_{\mathbf{h}(t)} = \boldsymbol{A}_{\mathbf{H}(t)} - \boldsymbol{A}_{\mathbf{H}_\infty}$ for $t \ge 0$, omitting the argument (t) if no confusion is possible. Note that the mapping $\boldsymbol{h} \mapsto \boldsymbol{A}_{\mathbf{h}}$

is linear, time independent, and continuous from $[W^{\theta,2}(\Omega)]^n$ to $[W^{1+\theta,2}(\Omega)]^n$ for $0 \leq \theta \leq 1$; see [1, Lemma 4].

We give ψ and $\boldsymbol{A}'$ a subscript to indicate whether they are components of u or v,

$$u = (\psi_u, \boldsymbol{A}'_u) = (\psi_u, \boldsymbol{A}_u - \boldsymbol{A}_{\mathbf{H}}), \quad v = (\psi_v, \boldsymbol{A}'_v) = (\psi_v, \boldsymbol{A}_v - \boldsymbol{A}_{\mathbf{H}_\infty}).$$

Thus, ψ_u and $\boldsymbol{A}_u$ satisfy Eqs. (1.10)–(1.12), ψ_v and $\boldsymbol{A}_v$ Eqs. (3.5)–(3.7). We denote the difference $w = u - v$ and use the same subscript convention for the components of w,

$$w = (\psi_w, \boldsymbol{A}'_w) = (\psi_u - \psi_v, \boldsymbol{A}'_u - \boldsymbol{A}'_v) = (\psi_u - \psi_v, \boldsymbol{A}_u - \boldsymbol{A}_v - \boldsymbol{A}_{\mathbf{h}}).$$

Subtracting Eq. (4.3) from Eq. (4.2), we find that w satisfies the initial-value problem

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \mathcal{A}w = \mathcal{H}(t, w(t)) \quad \text{for } t > s; \quad w(s) = u_s - v_s, \tag{4.4}$$

in $\mathcal{L}^2$, where

$$\mathcal{H}(t, w) = \mathcal{H}_1(t, u, v)w + \mathcal{H}_2(t, u, v)\boldsymbol{h} + (0, -\boldsymbol{A}_{\partial_t \mathbf{H}})\,. \tag{4.5}$$

The first term on the right-hand side is linear in the components of w,

$$\mathcal{H}_1(t, u, v)w = (\varphi_1(t, u, v)w, \boldsymbol{F}_1(u, v)w)\,,$$

where

$$\begin{aligned}
\varphi_1(t, u, v)w = \frac{1}{\eta}\Big[&-\frac{2i}{\kappa}\,[(\nabla\psi_w)\cdot(\boldsymbol{A}'_v + \boldsymbol{A}_{\mathbf{H}_\infty}) + (\nabla\psi_u)\cdot\boldsymbol{A}'_w] \\
&-\frac{i}{\kappa}\,(1 - \eta\kappa^2\omega)\,[\psi_w(\nabla\cdot\boldsymbol{A}'_v) + \psi_u(\nabla\cdot\boldsymbol{A}'_w)] \\
&-\psi_w|\boldsymbol{A}'_v + \boldsymbol{A}_{\mathbf{H}_\infty}|^2 - \psi_u\boldsymbol{A}'_w\cdot(\boldsymbol{A}'_u + \boldsymbol{A}'_v + \boldsymbol{A}_{\mathbf{H}} + \boldsymbol{A}_{\mathbf{H}_\infty}) \\
&+\left(1 - |\psi_u|^2 - |\psi_v|^2\right)\psi_w - \psi_u\psi_v\psi_w^*\Big]\,, \\
\boldsymbol{F}_1(u, v)w = \frac{1}{2i\kappa}\,&[\psi_u^*\nabla\psi_w + \psi_w^*\nabla\psi_v - \psi_u\nabla\psi_w^* - \psi_w\nabla\psi_v^*] \\
&-(\psi_w\psi_v^* + \psi_u\psi_w^*)(\boldsymbol{A}'_v + \boldsymbol{A}_{\mathbf{H}_\infty}) - |\psi_u|^2\boldsymbol{A}'_w.
\end{aligned}$$

(The explicit dependence of φ_1 on t is caused by the term $\boldsymbol{A}_{\mathbf{H}}$.) The second term on the right-hand side of Eq. (4.5) is linear in $\boldsymbol{h}$,

$$\mathcal{H}_2(t, u, v)\boldsymbol{h} = (\varphi_2(t, u, v)\boldsymbol{h}, \boldsymbol{F}_2(u)\boldsymbol{h})\,,$$

where

$$\varphi_2(t,u,v)\boldsymbol{h} = \frac{1}{\eta}\left[-\frac{2i}{\kappa}\nabla\psi_u - \psi_u(\boldsymbol{A}'_u + \boldsymbol{A}'_v + \boldsymbol{A}_{\mathbf{H}} + \boldsymbol{A}_{\mathbf{H}_\infty})\right]\cdot\boldsymbol{A}_{\mathbf{h}},$$

$$\boldsymbol{F}_2(u)\boldsymbol{h} = -|\psi_u|^2\boldsymbol{A}_{\mathbf{h}}.$$

The last term on the right-hand side of Eq. (4.5) accounts for the time dependence of $\boldsymbol{H}$; it is linear in $\partial_t\boldsymbol{H}$.

Let B_R be the ball of radius R centered at the origin in $\mathcal{W}^{1+\alpha,2}$. We claim that the mapping $w \mapsto \mathcal{H}_1(t,u,v)w : \mathcal{W}^{1+\alpha,2} \to \mathcal{L}^2$ is uniformly bounded for all $t \geq s$ and all $u, v \in B_R$. The claim is proved by estimating each term in $\mathcal{H}_1(t,u,v)w$ separately. For example,

$$\|\psi_u^*\nabla\psi_w\|_{L^2} \leq \|\psi_u\|_{L^\infty}\|\psi_w\|_{W^{1,2}} \leq C\|w\|_{\mathcal{W}^{1+\alpha,2}},$$

where C is a positive constant, which depends only on R. Similar estimates hold for the other terms, so

$$\|\mathcal{H}_1(t,u,v)w\|_{\mathcal{L}^2} \leq C\|w\|_{\mathcal{W}^{1+\alpha,2}} \quad \text{for } t \geq s, \tag{4.6}$$

for all u and v in B_R, where C is a positive constant, which depends on R, but not on s or t. The norms of $\mathcal{H}_2(t,u,v)\boldsymbol{h}$ and $\boldsymbol{A}_{\partial_t\mathbf{H}}$ are readily estimated,

$$\|\mathcal{H}_2(t,u,v)\boldsymbol{h}\|_{\mathcal{L}^2} \leq C\|\boldsymbol{h}\|_{W^{\alpha,2}} \quad \text{for } t \geq s, \tag{4.7}$$

$$\|\boldsymbol{A}_{\partial_t\mathbf{H}}\|_{W^{1,2}} \leq C\|\partial_t\boldsymbol{H}\|_{L^2} \quad \text{for } t \geq s, \tag{4.8}$$

where, again, the positive constants C depend on R, but not on s or t.

Take any $t \geq s$. From Eqs. (4.4) and (4.5) we obtain the inequality

$$\begin{aligned}\|w(t)\|_{\mathcal{W}^{1+\alpha,2}} &\leq \|\mathrm{e}^{-\mathcal{A}(t-s)}(u_s - v_s)\|_{\mathcal{W}^{1+\alpha,2}} \\ &+ \int_s^t \left\|\mathrm{e}^{-\mathcal{A}(t-t')}\mathcal{H}_1(t',u(t'),v(t'))w(t')\right\|_{\mathcal{W}^{1+\alpha,2}}\,\mathrm{d}t' \\ &+ \int_s^t \left\|\mathrm{e}^{-\mathcal{A}(t-t')}\mathcal{H}_2(t',u(t'),v(t'))\boldsymbol{h}(t')\right\|_{\mathcal{W}^{1+\alpha,2}}\,\mathrm{d}t' \\ &+ \int_s^t \left\|\mathrm{e}^{-\mathcal{A}(t-t')}\,(0,-\boldsymbol{A}_{\partial_t\mathbf{H}}(t'))\right\|_{\mathcal{W}^{1+\alpha,2}}\,\mathrm{d}t'.\end{aligned}$$

Keeping in mind the inequality (4.1), we apply the estimates (4.6)–(4.8) and conclude that

$$\begin{aligned}\|w(t)\|_{\mathcal{W}^{1+\alpha,2}} &\leq C_1\,\mathrm{e}^{-\lambda_1(t-s)}\,\|u_s - v_s\|_{\mathcal{W}^{1+\alpha,2}} \\ &+ C_2\int_s^t \max\{(t-t')^{-(1+\alpha)/2},1\}\,\mathrm{e}^{-\lambda_1(t-t')}\,\|w(t')\|_{\mathcal{W}^{1+\alpha,2}}\,\mathrm{d}t'\end{aligned}$$

$$+C_3 \int_s^t \max\{(t-t')^{-(1+\alpha)/2}, 1\}\ \mathrm{e}^{-\lambda_1(t-t')}\ \|\boldsymbol{h}(t')\|_{W^{\alpha,2}}\, \mathrm{d}t'$$
$$+C_4 \int_s^t \max\{(t-t')^{-(1+\alpha)/2}, 1\}\ \mathrm{e}^{-\lambda_1(t-t')}\ \|\partial_t \boldsymbol{H}(t')\|_{L^2}\, \mathrm{d}t'. \tag{4.9}$$

Here, C_1 through C_4 are positive constants, which depend on R, but not on s or t.

To obtain an upper bound for $\|w(t)\|_{\mathcal{W}^{1+\alpha,2}}$, $t \geq s$, we take a number $\Lambda > 0$ to be determined later and define the function f on $[s, \infty)$ by the expression

$$f(t) = \sup_{t' \in [s,t]} \left(\|w(t')\|_{\mathcal{W}^{1+\alpha,2}}\ \mathrm{e}^{-(\Lambda-\lambda_1)t'} \right) \quad \text{for } t \geq s.$$

Given any number $\gamma \in (0,1)$, we also introduce the convolution kernel

$$k_\gamma(s') = \max\{(s')^{-\gamma}, 1\} \mathrm{e}^{-(\Lambda-\lambda_1)s'} \quad \text{for } s' \geq 0.$$

Then there exists, for every p with $1 \leq p < 1/\gamma$, a positive constant $C_{\gamma,p}$, which does not depend on Λ ($\Lambda > 0$), such that

$$\left(\int_0^\infty k_\gamma(s')^p\, \mathrm{d}s' \right)^{1/p} \leq C_{\gamma,p} \Lambda^{\gamma-(1/p)} \quad \text{for all } \Lambda > 0. \tag{4.10}$$

Applying Hölder's inequality to the various integrals in (4.9) and using the inequality (4.10), we obtain the estimate

$$f(t) \leq C_1 \|u_s - v_s\|_{\mathcal{W}^{1+\alpha,2}}\ \mathrm{e}^{-(\Lambda-\lambda_1)s}$$
$$+\Lambda^{-(1-\alpha)/2} \left\{ C_{(1+\alpha)/2,1} \left[C_2 f(t) + C_3 \sup_{t' \in [s,t]} \left(\|\boldsymbol{h}(t')\|_{\mathcal{W}^{\alpha,2}}\ \mathrm{e}^{-(\Lambda-\lambda_1)t'} \right) \right] \right.$$
$$\left. +C_4 C_{\alpha/2,2} \left(\int_s^t \|\partial_t \boldsymbol{H}(t')\|_{L^2}^2\ \mathrm{e}^{-2(\Lambda-\lambda_1)t'}\, \mathrm{d}t' \right)^{1/2} \right\} \quad \text{for } t \geq s. \tag{4.11}$$

We take $\Lambda > 0$ sufficiently large that $\Lambda \geq \lambda_1$ and $C_2 C_{(1+\alpha)/2,1} \Lambda^{-(1-\alpha)/2} \leq \frac{1}{2}$. Then it follows from the inequality (4.11) that

$$f(t) \leq C \left[\|u_s - v_s\|_{\mathcal{W}^{1+\alpha,2}}\ \mathrm{e}^{-(\Lambda-\lambda_1)s} + \sup_{t' \in [s,t]} \left(\|\boldsymbol{h}(t')\|_{\mathcal{W}^{\alpha,2}}\ \mathrm{e}^{-(\Lambda-\lambda_1)t'} \right) \right.$$
$$\left. + \left(\int_s^t \|\partial_t \boldsymbol{H}(t')\|_{L^2}^2\ \mathrm{e}^{-2(\Lambda-\lambda_1)t'}\, \mathrm{d}t' \right)^{1/2} \right] \quad \text{for } t \geq s,$$

where the positive constant C depends on R, but not on s or t. This estimate can be rewritten as

$$\|w(t)\|_{\mathcal{W}^{1+\alpha,2}} \leq C \left[\|u_s - v_s\|_{\mathcal{W}^{1+\alpha,2}}\ \mathrm{e}^{(\Lambda-\lambda_1)(t-s)} \right.$$

$$+ \sup_{t' \in [s,t]} \left(\|\boldsymbol{h}(t')\|_{\mathcal{W}^{\alpha,2}} \, \mathrm{e}^{(\Lambda-\lambda_1)(t-t')} \right)$$

$$+ \left(\int_s^t \|\partial_t \boldsymbol{H}(t')\|_{L^2}^2 \, \mathrm{e}^{2(\Lambda-\lambda_1)(t-t')} \, \mathrm{d}t' \right)^{1/2} \Bigg] \quad \text{for } t \geq s. \tag{4.12}$$

Hence,

$$\|w(t)\|_{\mathcal{W}^{1+\alpha,2}} \leq C\mathrm{e}^{(\Lambda-\lambda_1)(t-s)} \Bigg[\|u_s - v_s\|_{\mathcal{W}^{1+\alpha,2}} + \sup_{t' \geq s} \|\boldsymbol{h}(t')\|_{\mathcal{W}^{\alpha,2}}$$

$$+ \left(\int_s^\infty \|\partial_t \boldsymbol{H}(t')\|_{L^2}^2 \, \mathrm{d}t' \right)^{1/2} \Bigg] \quad \text{for } t \geq s. \tag{4.13}$$

The desired inequality (3.16) follows from (4.13) if we choose simultaneously $\delta > 0$ sufficiently small and $t_0 \geq 0$ sufficiently large that

$$\delta + \sup_{t' \geq t_0} \|\boldsymbol{h}(t')\|_{\mathcal{W}^{\alpha,2}} + \left(\int_{t_0}^\infty \|\partial_t \boldsymbol{H}(t')\|_{L^2}^2 \, \mathrm{d}t' \right)^{1/2} \leq \frac{\varepsilon}{C}.$$

Hypotheses **(H4')** and **(H4")** guarantee that such a choice is possible. ∎

4.2 Proof of Theorem 2

Proof. (i) See [1, Theorem 3(i)].

(ii) See [1, Lemma 1].

(iii) See [1, Theorem 3(ii)] for S. It remains to prove (iii) for U.

The functional $E_{\omega,\mathbf{H}(t)}[\psi, \boldsymbol{A}]$ defined in Eq. (3.20) is coercive on $\mathcal{W}^{1,2}$; see [5, Chapter I, Eq. (5.45)]. Given a weak solution $(\psi, \boldsymbol{A})$ of the gauged TDGL equations, we let $E_\omega(t) \equiv E_{\omega,\mathbf{H}(t)}[\psi(t), \boldsymbol{A}(t)]$. The function E_ω is bounded on $[0, \infty)$, because of the inequality (3.21) and the hypothesis **(H4")**. Its coercivity property then implies

$$\psi \in L^\infty(0, T; [W^{1,2}(\Omega)]^2) \quad \text{and} \quad \boldsymbol{A} \in L^\infty(0, T; [W^{1,2}(\Omega)]^n).$$

Also, $\boldsymbol{A}_\mathbf{H} \in L^\infty(0, \infty; [W^{1,2}(\Omega)]^n)$, because of the hypothesis **(H4')**. Hence, $u = (\psi, \boldsymbol{A}') \in L^\infty(0, \infty; \mathcal{W}^{1,2})$, which shows the boundedness of the orbit of each $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ in $\mathcal{W}^{1,2}$.

We improve this regularity result by taking advantage of the smoothing action of the semigroup $\mathrm{e}^{-\mathcal{A}t}$; see the proof of global existence in [1, Theorem 1].

We first treat $\boldsymbol{A}'$ and then use the result to improve the regularity of ψ. Each term in $\boldsymbol{F}$, which has been defined in Eq. (2.9), needs to be estimated separately. The smoothing action of $\mathrm{e}^{-\mathcal{A}t}$ applied to the term $\partial_t \boldsymbol{A}_{\mathbf{H}}$ yields the integral

$$\mathcal{J}_{\mathbf{H}}(t) = \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)} \left(0, \boldsymbol{A}_{\partial_t \mathbf{H}}(s)\right) \mathrm{d}s.$$

Making use of hypothesis **(H4")**, one shows that $\mathcal{J}_{\mathbf{H}} : [0, \infty) \mapsto [W^{1+\alpha,2}(\Omega)]^n$ is a bounded continuous function; see [1, Lemma 5]. The remaining terms in $\boldsymbol{F}$ are estimated in a standard way. For example,

$$\|\psi^* \nabla \psi\|_{L^2} \leq \|\psi\|_{L^\infty} \|\psi\|_{W^{1,2}} \leq C \|u\|_{\mathcal{W}^{1,2}}.$$

Here, $C = \max\{1, \|\psi_0\|_{L^\infty}\}$, which is independent of ψ. Similar estimates hold for the other terms in $\boldsymbol{F}$, so $\boldsymbol{F} \in L^\infty(0, \infty; [L^2(\Omega)]^n)$. Therefore,

$$\left(t \mapsto \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)} \boldsymbol{F}(s) \, \mathrm{d}s\right) \in L^\infty(0, \infty; [W^{1+\alpha,2}(\Omega)]^n),$$

so $\boldsymbol{A}' \in L^\infty(0, \infty; [W^{1+\alpha,2}(\Omega)]^n)$.

Next, we improve the regularity of ψ, which has been defined in Eq. (2.8). Again, each term in φ needs to be estimated separately. For example,

$$\|(\nabla \psi) \cdot (\boldsymbol{A}_{\mathbf{H}} + \boldsymbol{A}')\|_{L^2} \leq \|(\nabla \psi) \cdot \boldsymbol{A}_{\mathbf{H}}\|_{L^2} + \|(\nabla \psi) \cdot \boldsymbol{A}'\|_{L^2},$$

where

$$\|(\nabla \psi) \cdot \boldsymbol{A}_{\mathbf{H}}\|_{L^2} \leq \|\nabla \psi\|_{L^2} \|\boldsymbol{A}_{\mathbf{H}}\|_{L^\infty} \leq C \|u\|_{\mathcal{W}^{1,2}} \|\boldsymbol{A}_{\mathbf{H}}\|_{W^{1+\alpha,2}}$$

and

$$\|(\nabla \psi) \cdot \boldsymbol{A}'\|_{L^2} \leq \|\nabla \psi\|_{L^2} \|\boldsymbol{A}'\|_{L^\infty} \leq C \|u\|_{\mathcal{W}^{1,2}} \|\boldsymbol{A}'\|_{W^{1+\alpha,2}}.$$

(To obtain the last estimate, we used the Sobolev imbedding theorem.) Similar estimates hold for the other terms in φ, so $\varphi \in L^\infty(0, \infty; [L^2(\Omega)]^2)$, and, therefore, $\psi \in L^\infty(0, \infty; [W^{1+\alpha,2}(\Omega)]^2)$. Hence, $u = (\psi, \boldsymbol{A}') \in L^\infty(0, \infty; \mathcal{W}^{1+\alpha,2})$, which shows the boundedness of the orbit of each $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ in $\mathcal{W}^{1+\alpha,2}$.

The compactness of the orbit closure is an immediate consequence of the boundedness and [1, Corollary 2].

(iv) See [1, Theorem 3(iii)] for S. It remains to prove (iv) for U.

Let $s \geq 0$ be fixed. It follows from (iii) that the omega-limit set $\omega_s(u_0)$ of each $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ under $U(s+t, s)$, $t \geq 0$, is nonempty and compact. We prove by contradiction that $\omega_s(u_0)$ is connected.

Suppose $\omega_s(u_0)$ is not connected. Then $\omega_s(u_0) = K_1 \cup K_2$, where K_1 and K_2 are compact and disjoint. Hence, there exist two disjoint open neighborhoods N_1 and N_2 of K_1 and K_2, respectively, in $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ and $t_0 \geq 0$, such that $U(s+t,s)u_0 \in N_1 \cup N_2$ for all $t \geq t_0$. But $\{U(s+t,s)u_0 : t \geq t_0\}$, being the image of the interval $[t_0, \infty)$, is connected, so we have a contradiction.

The proof that the omega-limit set $\omega_s(u_0)$ of $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ consists exclusively of equilibrium points for S parallels [7, Chapter VII, proof of Theorem 4.1]. We denote $u(t) = U(s+t,s)u_0$ and introduce the function E,

$$E(t) = \left(E^{1/2}_{\omega,\mathbf{H}(t)} - P(t)\right)^2 \quad \text{for } t \geq 0, \tag{4.14}$$

where $P(t)$ is given by Eq. (3.22). Equation (3.21) implies that $E(s+t) \leq E(s)$ for $s \geq 0$ and $t \geq 0$. Moreover, the limit $E_\infty = \lim_{t\to\infty} E(t)$ exists and is finite, by hypothesis **(H4")**. Next, we take advantage of the continuity of the Liapunov functional $(t,v) \mapsto E^{1/2}_{\omega,\mathbf{H}(t)}[v]$ and $t \mapsto P(t)$, combine it with Corollary 1, and let $s \to \infty$. Thus, for all $w = (\psi, \boldsymbol{A} - \boldsymbol{A}_\mathbf{H}) \in \omega_s(u_0)$ and $t \geq 0$, we have

$$E_{\omega,\mathbf{H}_\infty}[S(t)w] = E_{\omega,\mathbf{H}_\infty}[w] = (E^{1/2}_\infty + P_\infty)^2,$$

where $P_\infty = \lim_{t\to\infty} P(t) < \infty$. The remainder of the proof is standard; see [7, Chapter VII, proof of Theorem 4.1].

If $w = (\psi, \boldsymbol{A} - \boldsymbol{A}_\mathbf{H}) \in \omega_s(u_0)$, then $E_\omega[S(t)w] = E_\omega[w]$ for all $t > 0$, and the same argument as in [1, proof of Theorem 3(iii)] leads to the conclusion that $\omega(\nabla \cdot \boldsymbol{A}) = 0$ in Ω. Because $\omega > 0$, it follows that $\nabla \cdot \boldsymbol{A} = 0$. ∎

4.3 Proof of Theorem 3

Proof. We take an arbitrary open bounded set B in $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ satisfying $\mathcal{A}_U \cup \mathcal{A}_S \subset B$. The proof of the identity $\mathcal{A}_U = \mathcal{A}_S$ consists of two parts.

(i) $\mathcal{A}_S \subset \mathcal{A}_U$. From Eq. (3.18) we deduce that

$$S(t)B \subset \overline{\bigcup_{s\geq 0} U(s+t,s)B} \quad \text{for every } t \geq 0.$$

Therefore,

$$\bigcup_{t\geq t_0} S(t)B \subset \bigcup_{t\geq t_0} \overline{\bigcup_{s\geq 0} U(s+t,s)B} \subset \overline{\bigcup_{s\geq 0,\, t\geq t_0} U(s+t,s)B},$$

for every $t_0 \geq 0$, so $\omega(B;S) \subset \omega(B;U)$ for any open bounded set B in $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$. It follows that $\mathcal{A}_S \subset \mathcal{A}_U$.

(ii) $\mathcal{A}_U \subset \mathcal{A}_S$. Let $B_U = \bigcup_{0 \leq s \leq t < \infty} U(t,s)B$. The coercivity on $\mathcal{W}^{1,2}$ of the functional $E_{\omega,\mathbf{H}(t)}[\psi, \boldsymbol{A}]$ defined in Eq. (3.20) and the inequality (3.21) imply that B_U is bounded in $\mathcal{W}^{1+\alpha,2}$. Furthermore, [1, Corollary 2] implies that, for $0 \leq s < t < \infty$, each $U(t,s) : \mathcal{D}(\mathcal{A}^{(1+\alpha)/2}) \to \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ maps bounded sets into relatively compact sets. Then we have, for any positive number r,

$$\begin{aligned}\omega(B;U) &= \bigcap_{t_0 \geq 0} \overline{\bigcup_{s \geq 0,\, t \geq t_0} U(s+t+r,s)B} \\ &= \bigcap_{t_0 \geq 0} \overline{\bigcup_{s \geq 0,\, t \geq t_0} U(s+t+r,s+t)U(s+t,s)B} \\ &\subset \bigcap_{t_0 \geq 0} \overline{\bigcup_{s \geq 0,\, t \geq t_0} U(s+t+r,s+t)B_U} = \bigcap_{t_0 \geq 0} \overline{\bigcup_{s \geq t_0} U(s+r,s)B_U}. \end{aligned} \tag{4.15}$$

The last set coincides with the closure of $S(r)B_U$ in $\mathcal{W}^{1+\alpha,2}$ according to Eq. (3.18), so $\omega(B;U) \subset \overline{S(r)B_U}$ for every $r > 0$. It follows that $\mathcal{A}_U \subset \mathcal{A}_S$. ∎

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